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On the acyclicity of the Tate complex

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Abstract

We give a characterization of the acyclicity of the second step of a Tate or simplicial resolution of a commutative algebra. As a consequence, we obtain a characterization of the vanishing of André–Quillen cohomology of a surjective homomorphism $\phi : A \to B$, $H^j(A, B, -) = 0$ for all $j \geq 3$, in terms of the Koszul homology of $\mathrm{Ker}(\phi)$. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

All rings considered in this paper are commutative and unitary. Let A be a ring, I an ideal of A, $\boldsymbol{t} = \{t_\alpha,\ \alpha \in \Lambda\}$ a set of generators of I, $B = A/I$, and E the Koszul complex associated to $\boldsymbol{t}$. In the notation of [5, 10]:

$$E = E(\boldsymbol{t}) = A\langle T_\alpha;\ \mathrm{d}T_\alpha = t_\alpha,\ \alpha \in \Lambda\rangle.$$

Let $\boldsymbol{s} = \{s_v,\ v \in V\} \subset Z_1(E)$ be a set of representatives of a generating system of the B-module $H_1(E)$ and let

$$F = F(\boldsymbol{t}, \boldsymbol{s}) = E(\boldsymbol{t})\langle S_v;\ \mathrm{d}S_v = s_v,\ v \in V\rangle.$$

This differential graded algebra is the second step of a Tate resolution of B over A [10, Theorem 1; 5, Theorem 1.2.3]. We shall call F the Tate complex associated to $\boldsymbol{t}$ and $\boldsymbol{s}$.

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In [4, Theorem 2] the following result appears (which in the case A is a noetherian local ring and B its residue field, has already been proved in [3]):

Theorem 1. *The following conditions are equivalent*:

(i) $F(\boldsymbol{t},\boldsymbol{s})$ *is acyclic.*

(ii) *The set* $\boldsymbol{s}$ *represents a basis of the B-module* $H_1(E)$ *and the canonical homomorphism* $\gamma:\bigwedge_B H_1(E)\to H(E)$ *is an isomorphism.*

The proof of (ii)$\Rightarrow$(i) follows easily using a result of Tate [10, Theorem 2]. However, the proof of (i)$\Rightarrow$(ii) given in [4] is difficult and uses simplicial methods. The first purpose of the present paper is to give an alternative and easier proof of this result. For this, we shall construct first a spectral sequence relating $H(E)$ and $H(F)$.

After proving Theorem 1, we consider a similar question in simplicial theory, which has also been studied in [4]. With the previous notation, let X be a simplicial resolution of the A-algebra B obtained by a "step-by-step" construction, beginning with elements t_α and adjoining variables T_α to construct the first step K. If $\tilde{K}$ is the differential graded algebra associated to K, there exists a homomorphism $\phi:E\to\tilde{K}$ inducing isomorphisms in homology [4, Proposition 12]. Let L be the second step constructed adjoining variables S_v in degree 2 to kill a set of representatives s_v of a generating system of the B-module $\pi_1(K)=H_1(\tilde{K})$. In [4, Theorem 13] it is proved:

Theorem 2. *The following conditions are equivalent*:

(i) L *is acyclic, i.e.,* $\pi(L)=B$.

(ii) *The set* $\boldsymbol{s}$ *represents a basis of the B-module* $\pi_1(K)$ *and the canonical homomorphism* $\gamma:\bigwedge_B \pi_1(K)\to\pi(K)$ *is an isomorphism.*

In this paper we give a new proof of Theorem 2, analogous to the one of Theorem 1, constructing previously for it a spectral sequence relating $\pi(K)$ and $\pi(L)$.

We also see that both spectral sequences are isomorphic, and deduce from it that the canonical homomorphism of differential graded algebras $F\to\tilde{L}$ induces isomorphisms in homology, so giving an answer in the affirmative to a question in [4, Remark 14].

Finally, using Theorem 2 we obtain an alternative proof of the following result in [8] for the André–Quillen cohomology:

Corollary 3. *The following are equivalent*:

(i) $H^j(A,B,-)=0$ *for all* $j\geq 3$.

(ii) *The B-module* $H_1(E)$ *is projective and the canonical homomorphism* $\gamma:\bigwedge_B H_1(E)\to H(E)$ *is an isomorphism.*

Note that the proofs of Theorems 1 and 2 given in [4, Theorems 2 and 13] and the one of Corollary 3 given in [8, Corollary 3] are at present incomplete (see [9]), since it was used as a result still not proved in full generality: the existence of divided powers in the Künneth spectral sequence of simplicial algebras over a simplicial ring.

So the proofs of the present paper are not only alternative proofs but, in fact, the first complete proofs of the mentioned results.

2. Proofs

For the algebra F we have

$$F = \bigwedge_A E_1 \bigotimes_A \Gamma_A P,$$

where E_1 is the free A-module on the basis $\{T_\alpha,\ \alpha \in \Lambda\}$ and P is the free A-module with basis $\{S_v,\ v \in V\}$ in degree 2. The differential $\mathrm{d}: F \to F$ is the unique derivation such that

$$\mathrm{d}(T_\alpha \otimes 1) = t_\alpha, \quad \mathrm{d}(1 \otimes S_v^{(j)}) = s_v \otimes S_v^{(j-1)}.$$

Consider the bicomplex of A-modules defined by

$$M_{p,q} = \bigwedge_A^{p-q} E_1 \bigotimes_A \Gamma_A^q P,$$

with horizontal differential d' such that $\mathrm{d}'(T_\alpha \otimes 1) = t_\alpha$, $\mathrm{d}'(1 \otimes S_v^{(j)}) = 0$, and vertical differential d'' such that $\mathrm{d}''(T_\alpha \otimes 1) = 0$, $\mathrm{d}''(1 \otimes S_v^{(j)}) = s_v \otimes S_v^{(j-1)}$. Then $\mathrm{Tot}(M) = F$, and the spectral sequence associated to the second filtration of $\mathrm{Tot}(M)$ converges to $H(F)$:

$$E^1_{p,q} = H_{q-p}(E) \bigotimes_B \Gamma_B^p \left(B \bigotimes_A P \right) \Rightarrow H(F), \quad \mathrm{d}^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

Consider now the canonical homomorphism

$$\gamma : \bigwedge_B H_1(E) \to H(E).$$

For each $q \geq 0$ we have a morphism of complexes

$$\gamma_{q-*} \otimes 1 : \mathrm{Kos}^{q-*}(\psi)_q = \bigwedge_B^{q-*} H_1(E) \bigotimes_B \Gamma_B^* \left(B \bigotimes_A P \right)$$

$$\to H_{q-*}(E) \bigotimes_B \Gamma_B^* \left(B \bigotimes_A P \right) = E^1_{*,q},$$

where ψ is the epimorphism $B \otimes_A P \to H_1(E)$, $1 \otimes S_v \mapsto [s_v]$, and $\mathrm{Kos}^*(\psi)$ is the Koszul complex of ψ [6, Proposition I.4.3.1.2, p. 108].

The homomorphism ψ is an isomorphism if and only if s represents a basis of the B-module $H_1(E)$. In this case we have [6, Proposition I.4.3.1.6]

$$H_p(\mathrm{Kos}^{q-*}(\psi)_q) = H^{q-p}(\mathrm{Kos}^*(\psi)_q) = \begin{cases} B & \text{if } (p,q) = (0,0), \\ 0 & \text{if } (p,q) \neq (0,0). \end{cases}$$

Proof of Theorem 1. (ii)$\Rightarrow$(i). Since γ is an isomorphism we have an isomorphism of complexes $\mathrm{Kos}^{q-*}(\psi)_q \xrightarrow{\sim} E^1_{*,q}$, $q \geq 0$.

By the formula above,

$$E^2_{p,q} = H_p(\mathrm{Kos}^{q-*}(\psi)_q) = \begin{cases} B & \text{if } (p,q)=(0,0), \\ 0 & \text{if } (p,q)\neq(0,0). \end{cases}$$

Therefore, the spectral sequence collapses and we can deduce

$$H_n(F) = \begin{cases} B & \text{if } n=0, \\ 0 & \text{if } n\neq 0. \end{cases}$$

(i)$\Rightarrow$(ii). The epimorphism $B \otimes_A P \to H_1(E)$ identifies to the differential $\mathrm{d}^1 : E^1_{1,1} \to E^1_{0,1}$. From $E^1_{p,q}=0$ for $q<p$ and $H_2(F)=0$, we obtain $\ker(\mathrm{d}^1)=E^\infty_{1,1}=0$. So ψ is an isomorphism, i.e., s represents a basis of the B-module $H_1(E)$.

Next, we prove that γ_n is an isomorphism, $n\geq 0$, by induction on n. For $n=0,1$ it is clear. Assume $n \geq 1$ and γ_j isomorphism for $j \leq n$. The morphism of complexes

$$\gamma_{q-*}\otimes 1 : \mathrm{Kos}^{q-*}(\psi)_q \to E^1_{*,q}$$

verifies that $\gamma_{q-p}\otimes 1$ is an isomorphism for $q-p\leq n$. Thus,

$$H_p(\mathrm{Kos}^{q-*}(\psi)_q) \xrightarrow{\sim} E^2_{p,q} \quad \text{for } q-p+1 \leq n$$

and so

(I) $\qquad E^2_{p,q}=0 \quad \text{if } q-p+1\leq n, \ (p,q)\neq(0,0).$

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Kos}^{n-1}(\psi)_{n+1} \arrow[r, "\gamma_{n-1}\otimes 1"] \arrow[d] & E^1_{2,n+1} \arrow[d, "\mathrm{d}^1_{2,n+1}"] \\
\mathrm{Kos}^{n}(\psi)_{n+1} \arrow[r, "\gamma_n\otimes 1"] \arrow[d] & E^1_{1,n+1} \arrow[d, "\mathrm{d}^1_{1,n+1}"] \\
\mathrm{Kos}^{n+1}(\psi)_{n+1} \arrow[r, "\gamma_{n+1}\otimes 1"] \arrow[d] & E^1_{0,n+1} \arrow[d] \\
0 & 0,
\end{tikzcd}$$

where the left column is exact. From (I) we deduce $E^2_{0,n+1}=E^\infty_{0,n+1}, E^2_{1,n+1}=E^\infty_{1,n+1}$. So $E^2_{0,n+1}=E^2_{1,n+1}=0$ since $H_{n+1}(F)=H_{n+2}(F)=0$, and thus, the right column is also exact. Since γ_{n-1},γ_n are isomorphisms, so is γ_{n+1}. $\square$

Proof of Theorem 2. Consider the simplicial A-algebras K and L defined in the introduction, and the filtration of L

$$0 = L_m^{-1} \subset L_m^0 \subset L_m^1 \subset \cdots$$

given by the degree as polynomials over K_m:

$$L_m^k = \{f \in L_m / \deg(f) \leq k\}.$$

We shall prove that the associated spectral sequence

$$E^1_{p,q} = \pi_{p+q}(L^p/L^{p-1}) \Rightarrow \pi(L)$$

converges.

We have an isomorphism [2, p. 330]

$$L^k/L^{k-1} = K \bigotimes_A S_A^k(M),$$

where M is the simplicial A-module such that $M_0 = M_1 = 0$, $M_2 = \bigoplus_v AS_v$, $\varepsilon_2^i(S_v) = 0$, $0 \leq i \leq 2$, and for $h > 0, M_{2+h}$ is the free A-module with basis

$$\sigma_{1+h}^{i_1}\sigma_h^{i_2}\cdots\sigma_2^{i_h}(S_v), \quad 0 \leq i_h < \cdots < i_2 < i_1 \leq 1+h, \ \ v \in V.$$

Let $c(H_2(M))$ denote the constant simplicial A-module $H_2(M)$ and Σ the suspension functor. Using Quillen's isomorphisms [6, I.4.3.2.1], we see that

$$\begin{aligned} H_r(S_A^k M) &= H_r(S_A^k \Sigma\Sigma c(H_2(M))) = H_{r-2k}(\Gamma_A^k c(H_2(M))) \\ &= \begin{cases} 0 & \text{if } r \neq 2k, \\ \Gamma_A^k H_2(M) & \text{if } r = 2k. \end{cases} \end{aligned}$$

By Künneth formula we obtain

$$\pi_m(L^k/L^{k-1}) = 0 \quad \text{if } m \neq 2k,$$

and having in mind the exact sequence

$$\pi_j(L^k/L^{k-1}) \to \pi_j(L/L^{k-1}) \to \pi_j(L/L^k),$$

we deduce that

$$\pi_j(L/L^{k-1}) \to \pi_j(L/L^t)$$

is injective if $j < 2k$ and $t \geq k-1$. So

$$\pi_j(L/L^{k-1}) = 0 \quad \text{if } j < 2k$$

since an element $x \in \pi_j(L/L^{k-1})$ has a representative $f + L^{k-1}$, where $f \in L_j$ has some degree $t \geq k-1$. The image of x in $\pi_j(L/L^t)$ has then a representative $f + L^t = L^t$. Since $\pi_j(L/L^{k-1}) \to \pi_j(L/L^t)$ is injective, $x = 0$.

Now, from the exact sequence

$$\pi_{j+1}(L/L^{k-1}) \to \pi_j(L^{k-1}) \to \pi_j(L) \to \pi_j(L/L^{k-1})$$

we have that $\pi_j(L^{k-1}) \to \pi_j(L)$ is surjective if $j<2k$ and an isomorphism if $j<2k-1$.

This implies the convergence of the spectral sequence:

$$E^1_{p,q} = \pi_{q-p}(K) \bigotimes_B \Gamma^p_B \left(\bigoplus_v BS_v \right) \Rightarrow \pi(L).$$

Therefore, we can finish the proof of Theorem 2 as the one of Theorem 1. □

Now, we shall compare the two spectral sequences that we have constructed. It is known [4, Proposition 12] that there exists a homomorphism of differential graded A-algebras $\phi: E \to \tilde{K}$ inducing isomorphisms in homology. Since $\tilde{L}$ is a differential graded A-algebra with divided powers, there exist a unique homomorphism of differential graded A-algebras with divided powers $\vartheta: F \to \tilde{L}$ such that ϑ and ϕ agree on E and $\vartheta(S_v) = S_v$. Since ϑ is compatible with the filtrations, it induces a homomorphism between the two spectral sequences, which is an isomorphism in E^1. So both spectral sequences are isomorphic and so the homomorphism induced by ϑ

$$H(F) \to H(\tilde{L}) = \pi(L)$$

is an isomorphism.

The fact that ϑ is an isomorphism can also be deduced from a result of Pitteloud [7, Proposition 3.4] using that $H(E) \xrightarrow{\sim} H(\tilde{K})$, and it is the affirmative answer to the question in [4, Remark 14].

In order to prove Corollary 3 we will need the following:

Lemma 4. *The following conditions are equivalent*:

(i) *L is acyclic.*

(ii) *$H_j(A,B,B) = 0$ for all $j \geq 3$ and s represents a basis of the B-module $\pi_1(K)$.*

Proof. Analogous to the proof of [4, Proposition 11]. □

Proof of Corollary 3. By [8, Proposition 8] condition (i) of Corollary 3 is equivalent to: $H_j(A,B,B) = 0$ for all $j \geq 3$ and $H_1(E)$ is B-projective. Localizing at a prime ideal of A containing I, we have to prove that the following are equivalent:

(a) $H_j(A,B,B) = 0$ for all $j \geq 3$ and the B-module $H_1(E)$ is free.

(b) The canonical homomorphism $\gamma: \bigwedge_B H_1(E) \to H(E)$ is an isomorphism and the B-module $H_1(E)$ is free.

This equivalence is a consequence of Lemma 4, Theorem 2 and the isomorphism $H(E) \xrightarrow{\sim} H(\tilde{K})$. □

Finally, we deduce from Corollary 3 the first part of [8, Corollary 3] in the noetherian case.

Corollary 3′. *If A is noetherian, the following conditions are equivalent:*

(i) $H_j(A,B,-)=0$ *for all* $j\geq 3$.

(ii) *The B-module $H_1(E)$ is flat and the canonical homomorphism* $\gamma:\bigwedge_B H_1(E)\to H(E)$ *is an isomorphism.*

Proof. Having in mind [1, Proposition 4.57], it is enough to show that if the B-module $H_1(E)$ is flat then it is projective. This is clear if the set of generators $\{t_\alpha,\ \alpha\in\Lambda\}$ of I is finite, since in this case $H_1(E)$ is a B-module of finite type. In the general case, since I is of finite type, there exist elements $\alpha_1,\dots,\alpha_n\in\Lambda$ such that $\{t_{\alpha_1},\dots,t_{\alpha_n}\}$ generates I. For $E':=A\langle T_{\alpha_1},\dots,T_{\alpha_n};\ \mathrm{d}T_{\alpha_i}=t_{\alpha_i}\rangle$ it is easy to check that there exists a free B-module X such that $H_1(E')\oplus X=H_1(E)$ (alternatively it can be deduced from the following diagram of exact rows and columns, where the rows are the exact sequences of [1, Proposition 15.12]:

$$\begin{tikzcd}
0 \arrow[r] & H_2(A,B,B) \arrow[r] \arrow[d, equal] & H_1(E') \arrow[r] \arrow[d] & E_1'\otimes_A B \arrow[r] \arrow[d, tail, "\tau"] & I\otimes_A B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_2(A,B,B) \arrow[r] & H_1(E) \arrow[r] \arrow[d, two heads] & E_1\otimes_A B \arrow[r] \arrow[d, two heads] & I\otimes_A B \arrow[r] & 0 \\
 & & X \arrow[r, equal] & X & &
\end{tikzcd}$$

and where X is by definition the cokernel of τ, which is free). The result follows. □

Remark 5. Consider the filtration of K analogous to the one already defined for L, given by the degree as polynomials over A

$$K_m^r=\{f\in K_m/\deg(f)\leq r\}.$$

Reasoning as before we can prove that the associated spectral sequence

$$E^1_{p,q}=\pi_{p+q}(K^p/K^{p-1})\Rightarrow\pi(K)$$

converges and

$$E^1_{p,q}=\begin{cases}0 & \text{if } q\neq 0,\\ E_p & \text{if } q=0.\end{cases}$$

Therefore, the spectral sequence collapses on the q-axis and we have isomorphisms

$$\pi_n(K)=E^2_{n,0}=H_n(E).$$

This gives a new proof of [4, Proposition 12].

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